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Modeling and analysis of rotating solids and structures

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Summary. It is demonstrated that the inertial terms in the dynamic equations of motion in a moving frame take a simple universal form in terms of the classic mass matrix, when the traditional Lagrangian approach with local velocities as time derivatives of position is replaced by a Hamiltonian approach, in which absolute velocities and local positions are interpolated by identical shape functions. The resulting equations take on a simple systematic form that lends itself naturally to conservative time integration and permits a simple algorithmic damping scheme in terms of local motion.

Key words: Time integration, conservative integration, structural dynamics.

Introduction

Rotating structures are acted on by inertial forces generated by the rotation. If the local displacement of the structure at a generic point is represented by shape functions and local velocities are obtained by time differentiation of the interpolated displacements, the inertial effects from rotation leads to mass matrices containing the angular velocity and acceleration of the rotating frame of reference. While the inertial forces can be obtained via rearranging individual parts of the original mass matrix – see e.g. [1] – the appearance of separate representations of the inertial effects in the discretized system equations is computationally inconvenient and complicates the development of conservative time integration algorithms for rotating structures. This problem can be resolved by adopting a Hamiltonian view of mechanics, in which displacements and velocities (momentum) appear as independent variables, and therefore are interpolated separately from their nodal values. Hereby all inertial effects are represented by the classic mass matrix, and the effects of rotational convection are represented by global operations via vector products with the angular velocity of the rotating frame of reference.

The nodal displacements and velocities constitute a state-space representation of the local motion. When the local displacements are combined with the absolute velocities the corresponding hybrid state-space equations of motion take a particularly simple form without the angular acceleration and with the angular velocity only in linear form. A conservative integration algorithm for the hybrid state-space variables is obtained by using the mean value of the angular velocity over the current integration interval. For changing angular velocity the conservative integration format is different from the classic collocation format exemplified by the Newmark scheme. A damping based on local motion can be introduced by a simple modification of the coefficients of the state-space integration format.

Lagrange-Hamilton basics

Let a structure be described by a set of generalized coordinates $\boldsymbol{q} = [q_1, q_2, \cdots]^T$ with time derivative $\dot{\boldsymbol{q}} = [\dot{q}_1, \dot{q}_2, \cdots]^T$. The Lagrangian functional is then defined by

$$\mathcal{L} = \int_{t_1}^{t_2} L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\, dt \tag{1}$$

in terms of the Lagrangian density

$$L(\boldsymbol{q},\dot{\boldsymbol{q}},t) \;=\; T(\boldsymbol{q},\dot{\boldsymbol{q}},t) \;-\; V(\boldsymbol{q},t), \tag{2}$$

where $T(\boldsymbol{q},\dot{\boldsymbol{q}},t)$ is the kinetic energy, while $V(\boldsymbol{q},t)$ is a potential. The equations of motion follow from the variational condition $\delta\mathcal{L} = 0$, whereby

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\Big) \;-\; \frac{\partial L}{\partial \boldsymbol{q}} \;=\; \boldsymbol{0}. \tag{3}$$

In the Lagrangian formulation the variables are the generalized displacements $\boldsymbol{q}$, and the equations of motion are second order differential equations in these variables.

The Hamiltonian formulation starts by introducing the term in the parenthesis of (3) as the independent momentum,

$$\boldsymbol{p} \;=\; \frac{\partial L}{\partial \dot{\boldsymbol{q}}^T}. \tag{4}$$

The Hamiltonian $H(\boldsymbol{q},\boldsymbol{p},t)$ is then defined from the Lagrangian $L(\boldsymbol{q},\dot{\boldsymbol{q}},t)$ by the Legendre transform

$$H(\boldsymbol{q},\boldsymbol{p},t) \;=\; \dot{\boldsymbol{q}}^T\boldsymbol{p} \;-\; L(\boldsymbol{q},\dot{\boldsymbol{q}},t), \tag{5}$$

where the displacements $\boldsymbol{q}$ and the momentum $\boldsymbol{p}$ are treated as independent variables. The equations of motion then follow from stationarity in the classic form

$$\frac{d\boldsymbol{p}}{dt} \;=\; -\frac{\partial H}{\partial \boldsymbol{q}^T}\,, \qquad \frac{d\boldsymbol{q}}{dt} \;=\; \frac{\partial H}{\partial \boldsymbol{p}^T}. \tag{6}$$

The important observation in the present context is that by introducing the additional conjugate variable $\boldsymbol{p}$, the original set of second order differential equations are transformed into a set of first order equations in $\boldsymbol{q}$ and $\boldsymbol{p}$. The symmetric form of the Hamiltonian equations suggest representation of these independent variables by identical interpolation schemes.

Rotating structures

A rotating structure is illustrated in Fig. 1. The position of the nodes are described by the coordinates $\boldsymbol{x}_n$ in a frame of reference rotating with angular velocity $\boldsymbol{\Omega}$. A generic point with internal coordinate $\boldsymbol{\xi}$ is the given in the local frame in terms of the coordinates of the nodes as

$$\boldsymbol{x}_\xi \;=\; \underbrace{\boldsymbol{N}(\boldsymbol{\xi})}_{3\times N}\boldsymbol{x}_n. \tag{7}$$

The issue here is the representation of the corresponding velocity.

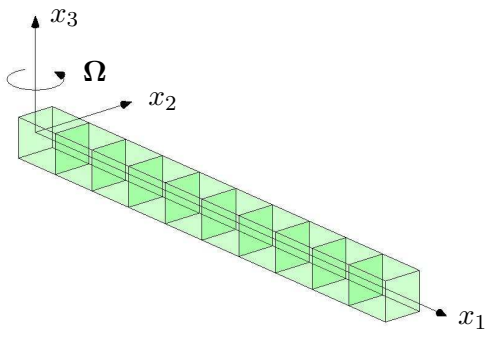

Figure 1. Solid body in frame $\{x_1, x_2, x_3\}$ rotating with angular velocity $\boldsymbol{\Omega}$.

Classic representation

The classic approach is Lagrangian in the sense that the local components of the global velocity $\boldsymbol{v}_\xi$ is obtained at the generic point $\boldsymbol{x}_\xi$ as an absolute time derivative, combining the local velocity and a convective velocity from the rotation of the frame of reference,

$$\boldsymbol{v}_\xi \;=\; D_t \boldsymbol{x}_\xi \;=\; (\partial_t + \tilde{\boldsymbol{\Omega}})\boldsymbol{x}_\xi, \tag{8}$$

where the notation $\tilde{\boldsymbol{\Omega}} = \boldsymbol{\Omega}\times$ is used for the skew symmetric matrix representing the vector product. The kinetic energy is given in terms of the absolute velocity as

$$T \;=\; \int_V \tfrac{1}{2}\rho\, \boldsymbol{v}_\xi^T \boldsymbol{v}_\xi \, dV_\xi. \tag{9}$$

The absolute velocity depends on the local nodal velocity $\dot{\boldsymbol{x}}_n$ and the local position $\boldsymbol{x}_n$ and substitution from (8) gives the discretized form

$$T \;=\; \tfrac{1}{2}[\dot{\boldsymbol{x}}_n^T, \boldsymbol{x}_n^T] \begin{bmatrix} \boldsymbol{M}_{00} & \boldsymbol{M}_{01} \\ \boldsymbol{M}_{10} & \boldsymbol{M}_{11} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{x}}_n \\ \boldsymbol{x}_n \end{bmatrix}, \tag{10}$$

where the block matrices $\boldsymbol{M}_{jk}$ are defined by

$$\boldsymbol{M}_{jk} \;=\; \int_V \rho\, \boldsymbol{N}(\boldsymbol{\xi})^T \big(\tilde{\boldsymbol{\Omega}}^T\big)^j \tilde{\boldsymbol{\Omega}}^k \boldsymbol{N}(\boldsymbol{\xi})\, dV_\xi. \tag{11}$$

It is seen that the angular velocity $\boldsymbol{\Omega}$ is imbedded inside the volume integral. For elements that are not based on identical interpolation of all three displacement components – essentially all non-isoparametric elements – moving the angular velocity vector outside the integral requires some measure of restructuring of the mass matrix. In cases with time-dependent angular velocity this formulation therefore involves reassembly of the inertial matrices containing $\boldsymbol{\Omega}$ as well as terms representing the time derivatives of these matrices.

Hamiltonian representation

In the Hamiltonian formulation the generalized displacement vector $\boldsymbol{x}$ is supplemented by the corresponding momentum vector, defined via (4). The present paper is concerned with formulations in which the mass matrix is constant. This class includes isoparametric elements and elements that can be constructed using a definition of generalized strains as a quadratic function of the generalized displacements. The latter group includes energy-consistent moderate-strain formulations of e.g. beams, plates and shallow shells. When the mass matrix is constant the momentum vector can be replaced by the nodal velocity vector. The absolute velocity at a generic point $\boldsymbol{\xi}$ then follows from the interpolation format (7) as

$$\boldsymbol{v}_\xi \;=\; \boldsymbol{N}(\boldsymbol{\xi})\boldsymbol{v}_n \;=\; \boldsymbol{N}(\boldsymbol{\xi}) D_t \boldsymbol{x}_n. \tag{12}$$

In this format the velocity interpolates the nodal values obtained via the convected differential operator $D_t = \lceil \partial_t + \tilde{\boldsymbol{\Omega}} \rfloor = \partial_t + \tilde{\boldsymbol{\Omega}}_D$, now extended to global form by defining the block diagonal matrix $\tilde{\boldsymbol{\Omega}}_D = \lceil \tilde{\boldsymbol{\Omega}}, \cdots, \tilde{\boldsymbol{\Omega}} \rfloor$. This format gives the kinetic energy in terms of the mass matrix as

$$T \;=\; \tfrac{1}{2}[\dot{\boldsymbol{x}}_n^T, \boldsymbol{x}_n^T] \begin{bmatrix} \boldsymbol{M} & \boldsymbol{M}\tilde{\boldsymbol{\Omega}}_D \\ \tilde{\boldsymbol{\Omega}}_D^T \boldsymbol{M} & \tilde{\boldsymbol{\Omega}}_D^T \boldsymbol{M}\tilde{\boldsymbol{\Omega}}_D \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{x}}_n \\ \boldsymbol{x}_n \end{bmatrix}, \tag{13}$$

and thus the inertial loads from convection are obtained from global operations on the assembled mass matrix $\boldsymbol{M}$.

Equations of motion

The equations of motion can be expressed in several different forms. While the particular form may be less important when discussing the exact continuous case, the choice of format is a central issue in the development of discrete time integration algorithms. In the following only the discretized form is used, represented in terms of the nodal values, and the subscript n is therefore omitted.

Lagrangian form with classic interpolation

The equations of motion in the second-order Lagrange format (3) follow directly from differentiation of the discretized kinetic energy (10) and a potential $V(\boldsymbol{x})$ giving external forces $\boldsymbol{f}$ and internal forces $\boldsymbol{g}(\boldsymbol{x})$. The result is the classic equation

$$\boldsymbol{M}_{00}\ddot{\boldsymbol{x}} + (\boldsymbol{M}_{01} - \boldsymbol{M}_{10})\dot{\boldsymbol{x}} - \boldsymbol{M}_{11}\boldsymbol{x} + \dot{\boldsymbol{M}}_{01}\boldsymbol{x} + \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{f}. \tag{14}$$

In the case of isoparametric elements the angular velocity may be applied to the assembled mass matrix, and the equation takes the somewhat more intuitive form

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + (\boldsymbol{M}\tilde{\boldsymbol{\Omega}}_D + \tilde{\boldsymbol{\Omega}}_D\boldsymbol{M})\dot{\boldsymbol{x}} + \tilde{\boldsymbol{\Omega}}_D\boldsymbol{M}\tilde{\boldsymbol{\Omega}}_D\boldsymbol{x} + \boldsymbol{M}\dot{\tilde{\boldsymbol{\Omega}}}_D + \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{f}. \tag{15}$$

The second term is the gyroscopic or Coriolis force, the third is the centrifugal force, and the fourth term is the effect of angular acceleration. Even this special form, where the angular velocity has been extracted to the global format, does not lend itself immediately to energy conserving time integration because angular acceleration appears directly in the equation and the angular velocity appears in quadratic as well as in linear form. This dynamic equation is typically integrated by classic collocation schemes of collocation type, se e.g. [2].

Hybrid state-space format

The hybrid state-space format appears naturally, when observing that for the present problem $\boldsymbol{p} = \boldsymbol{M}\boldsymbol{v}$, and thus a natural variable combination is the local displacement $\boldsymbol{x}$ and the global velocity $\boldsymbol{v}$. The hybrid state-space equations of motion then take the form, [3],

$$\begin{bmatrix} \boldsymbol{0} & \boldsymbol{M} \\ -\boldsymbol{M} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{u}} \\ \dot{\boldsymbol{v}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{g}(\boldsymbol{u}) + \tilde{\boldsymbol{\Omega}}_D\boldsymbol{M}\boldsymbol{v} \\ \boldsymbol{M}\tilde{\boldsymbol{\Omega}}_D^T\boldsymbol{u} + \mathbf{M}\mathbf{v} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ -\boldsymbol{M}\tilde{\boldsymbol{\Omega}}_D^T\boldsymbol{x}_0 \end{bmatrix}. \tag{16}$$

where the displacements have been introduced as the difference between the current and the initial position, $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{x}_0$. These equations appear as a simple generalization of the classic state-space equations, e.g. [4], augmented by two linear terms in the angular velocity $\boldsymbol{\Omega}$. The hybrid state-space equations lend themselves directly to energy conserving time integration and permit a simple monotonic algorithmic damping scheme. This format is easily extended to models in which rotations are represented in quadratic form in terms of the generalized displacements, [5].

Conservative time integration

Conservative time integration algorithms are typically obtained from a time integral of the state-space equations of motion – in the present case (16). For structures with constant mass matrix the fist term changes directly into a similar term with the time increments of the state-space variables, $\Delta\boldsymbol{u}$ and $\Delta\boldsymbol{v}$. The issues to be resolved lie in the second term, which now represents a 'mean value' over the time integration interval Δt. It has been demonstrated in [3] that conservation properties are attained when the angular velocity $\boldsymbol{\Omega}$ is represented by its algebraic mean value of the initial and final values of the integration interval $\bar{\boldsymbol{\Omega}}$. In most problems

involving rotating bodies or structures the stiffening effect from stresses due to the centrifugal load play an essential role in balancing the direct centrifugal load term $\tilde{\boldsymbol{\Omega}}_D^T \boldsymbol{M} \tilde{\boldsymbol{\Omega}}_D$ to a greater or lesser extent, and geometric stiffness is therefore an important aspect of the problem. For elements with a quadratic strain measure, as in the present case, the effect of the kinematic non-linearity can be accounted for by the following simple result for the representative mean value of the internal forces, [4],

$$\mathbf{g}(\mathbf{u})_* \;=\; \overline{\mathbf{g}(\mathbf{u})} \;-\; \tfrac{1}{4}\Delta\mathbf{K}^g\Delta\mathbf{u}, \tag{17}$$

where $\Delta\mathbf{K}^g$ is the increment of the geometric stiffness matrix over the time interval Δt. In spite of the fact that this term formally is a 'higher order term' it is important for consistency and accuracy in problems depending on geometric stiffness.

When these two results are incorporated, the integrated form of the state-space equations of motion take the form

$$\begin{bmatrix} \frac{1}{4}\Delta t\Delta\mathbf{K}^g & \boldsymbol{M} \\ -\boldsymbol{M} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{u} \\ \Delta\boldsymbol{v} \end{bmatrix} + \Delta t \begin{bmatrix} \overline{\boldsymbol{g}(\boldsymbol{u})} + \bar{\tilde{\boldsymbol{\Omega}}}_D \boldsymbol{M}\bar{\boldsymbol{v}} \\ \boldsymbol{M}\bar{\tilde{\boldsymbol{\Omega}}}_D^T\bar{\boldsymbol{u}} + \boldsymbol{M}\,\bar{\boldsymbol{v}} \end{bmatrix} = \Delta t \begin{bmatrix} \boldsymbol{f}_* \\ -\boldsymbol{M}\bar{\tilde{\boldsymbol{\Omega}}}_D^T\boldsymbol{x}_0 \end{bmatrix}. \tag{18}$$

A local form of the energy balance can be obtained by expressing the absolute velocity $\boldsymbol{v}$ in terms of the local velocity $\dot{\boldsymbol{u}}$ from (8). Multiplication of the hybrid state-space equations (18) by $[\Delta\boldsymbol{u}^T, \Delta\boldsymbol{v}^T]$ leads to the following energy increment equation, [5],

$$\Big[\tfrac{1}{2}\dot{\boldsymbol{u}}^T\boldsymbol{M}\dot{\boldsymbol{u}} - \tfrac{1}{2}\boldsymbol{x}^T(\tilde{\boldsymbol{\Omega}}_D^T\boldsymbol{M}\tilde{\boldsymbol{\Omega}}_D)\boldsymbol{x} + G(\boldsymbol{u})\Big]_n^{n+1} \;+\; \overline{\boldsymbol{v}^T\boldsymbol{M}(\Delta\tilde{\boldsymbol{\Omega}}_D\boldsymbol{x})} \;=\; \Delta\boldsymbol{u}^T\boldsymbol{f}_*, \tag{19}$$

where $G(\boldsymbol{u}) = \Delta\boldsymbol{u}^T\boldsymbol{g}(\boldsymbol{u})$ is the increment of the internal energy, and $\Delta\boldsymbol{u}^T\boldsymbol{f}_*$ defines the work of the external force. The mean value term gives a direct representation of the contribution from angular acceleration within the time increment.

Local algorithmic damping

It is often desirable to introduce dissipation – partly to represent actual damping in the structure, and partly to dissipate high-frequency response components that are above the Nyquist frequency limit for reproduction of a continuous signal by its time-discretized counterpart. A convenient way of identifying a suitable format for algorithmic damping is to identify a desirable form of its dissipative contribution D to the energy balance equation. It has been demonstrated that for a stationary structure a suitable dissipation function is a quadratic form in the increments of the state-space variables, $D = \frac{1}{2}(\Delta\dot{\boldsymbol{u}}^T\boldsymbol{M}\Delta\dot{\boldsymbol{u}} + \Delta\boldsymbol{u}^T\boldsymbol{K}\Delta\boldsymbol{u})$, where $\boldsymbol{K}$ is a representative value of the stiffness matrix, [4]. In the present context it is desirable to formulate the dissipation in terms of local velocity $\dot{\boldsymbol{u}}$ in order for a purely convective rotation to be undamped. Furthermore, as seen from the energy balance equation, the local stiffness is reduced by $\tilde{\boldsymbol{\Omega}}_D^T\boldsymbol{M}\tilde{\boldsymbol{\Omega}}_D$. Thus, a suitable form of the dissipation potential in the present case is, [5],

$$D = \tfrac{1}{2}\alpha\Big\{\Delta\dot{\boldsymbol{u}}^T\boldsymbol{M}\Delta\dot{\boldsymbol{u}} + \Delta\boldsymbol{u}^T(\boldsymbol{K} - \bar{\tilde{\boldsymbol{\Omega}}}_D^T\boldsymbol{M}\bar{\tilde{\boldsymbol{\Omega}}}_D)\Delta\boldsymbol{u}\Big\}. \tag{20}$$

The algorithm is formulated in terms of the absolute velocity $\boldsymbol{v}$. The local velocity $\dot{\boldsymbol{u}}$ is therefore eliminated in favor of the absolute velocity via the relation (8), whereby the dissipation potential takes the simple matrix form

$$D = \tfrac{1}{2}\alpha\left[\Delta\boldsymbol{u}^T, \Delta\boldsymbol{v}^T\right] \begin{bmatrix} \boldsymbol{K} & \bar{\tilde{\boldsymbol{\Omega}}}_D\boldsymbol{M} \\ \boldsymbol{M}\bar{\tilde{\boldsymbol{\Omega}}}_D^T & \boldsymbol{M} \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{u} \\ \Delta\boldsymbol{v} \end{bmatrix} \tag{21}$$

When introducing $-D$ on the right side of the total energy balance equation, it is seen that the matrix including the factor $\frac{1}{2}\alpha$ should be included in the first matrix of the conservative

equations of motion (18). Hereby the discretized hybrid state-space equations take the final form

$$\begin{bmatrix} \frac{1}{4}\Delta t\Delta \mathbf{K}^g + \frac{1}{2}\alpha\Delta t\boldsymbol{K} & (\boldsymbol{I} + \frac{1}{2}\alpha\Delta t\bar{\tilde{\boldsymbol{\Omega}}}_D)\boldsymbol{M} \\ \boldsymbol{M}(-\boldsymbol{I} + \frac{1}{2}\alpha\Delta t\bar{\tilde{\boldsymbol{\Omega}}}_D^T) & \frac{1}{2}\alpha\Delta t\boldsymbol{M} \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{u} \\ \Delta\boldsymbol{v} \end{bmatrix} + \Delta t \begin{bmatrix} \overline{\boldsymbol{g}(\boldsymbol{u})} + \bar{\tilde{\boldsymbol{\Omega}}}_D\boldsymbol{M}\bar{\boldsymbol{v}} \\ \boldsymbol{M}\bar{\tilde{\boldsymbol{\Omega}}}_D^T\bar{\boldsymbol{u}} + \boldsymbol{M}\bar{\boldsymbol{v}} \end{bmatrix} = \Delta t \begin{bmatrix} \boldsymbol{f}_* \\ -\boldsymbol{M}\bar{\tilde{\boldsymbol{\Omega}}}_D^T\boldsymbol{x}_0 \end{bmatrix}. \tag{22}$$

It is noted that the the damping terms proportional to parameter α contribute in the form of symmetric terms in the first matrix. The combination of terms in the upper left corner of this matrix also indicates that the contribution $\frac{1}{4}\Delta\mathbf{K}^g$ has the form of a damping term. As the increment of the geometric stiffness may change sign, omission of this term by a simplified integration of the non-linear internal forces would lad to oscillations as illustrated in [4].

The non-dimensional damping parameter α appearing in the equations of motion can be related asymptotically to the modal damping ratio ζ_k in the low-frequency regime, [6],

$$\zeta_k \sim \tfrac{1}{2}\alpha\,(\omega_k\Delta t) \tag{23}$$

Thus, for the lower modes algorithmic damping in terms of the scalar damping parameter α leads to damping proportional to the modal frequency ω_k.

Numerical solution

The numerical solution of the hybrid state-space equations (22) proceeds in a simple step-by-step manner. First the equations are reformulated by using he second equation to express the current velocity $\boldsymbol{v}_n$ in terms of the current displacement $\boldsymbol{u}_n$. This expression is used to eliminate the current velocity $\boldsymbol{v}_n$ from the first equation, which then takes the form of a modified non-linear quasi-static static problem, for which standard solution procedures are available. When $\boldsymbol{u}_n$ has been determined, the current velocity $\boldsymbol{v}_n$ is determined from the relation used to eliminate this variable in the first step. Convergence is usually good due to the presence of the inertial terms that typically exercise a stabilizing effect for small time steps. Details and examples may be found in [4], [3] and [5].

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